

# Supplementary Materials: Deep Graph Random Process for Relational-Thinking-Based Speech Recognition

## 1 Proof of Theorem 1

**Theorem 1.** Let  $\mathcal{N}(\mu, \sigma^2)$  denotes a Gaussian distribution with  $\mu < 1/2$ , and let  $\mathcal{B}(n, \lambda)$  denotes a Binomial distribution with  $n \rightarrow +\infty$  and  $\lambda \rightarrow 0$ , where  $n$  is increasing while  $\lambda$  is decreasing. There exists a real constant  $m$  such that if  $m = n\lambda$  and if we define:

$$\begin{aligned} f_1(x) &= \text{KL}(\mathcal{N}(x, x(1-x)) || \mathcal{N}(\mu, \sigma^2)) \\ f_2(x) &= \text{KL}(\mathcal{N}(x, x(1-x)) || \mathcal{N}(n\lambda, n\lambda(1-\lambda))) \\ f_2^* &= \min_x f_2(x), \text{ where } x \in (0, 1) \end{aligned}$$

we have that:  $f_1(x)$  attains its minimum on the interval  $(0, 1)$  and  $f_2(x) - f_2^*$  is bounded on the interval  $(0, \sqrt{2}/2 - 1/2)$ , with:

$$x = m = \frac{1 + l - \sqrt{1 + l^2}}{2}, \text{ where } l = \frac{2\sigma^2}{1 - 2\mu}$$

*Proof.* The derivative of the function  $f_1(x)$  over  $x$  can be written as:

$$f_1'(x) = x^2 - \left(1 + \frac{2\sigma^2}{1 - 2\mu}\right)x + \frac{\sigma^2}{1 - 2\mu}$$

We set it as 0 and solve for  $x$ , giving

$$x = \begin{cases} \frac{1+l-\sqrt{1+l^2}}{2} & \text{if } \mu < 1/2 \\ \frac{1+l+\sqrt{1+l^2}}{2} & \text{if } \mu > 1/2 \end{cases}, \text{ where } l = \frac{2\sigma^2}{1 - 2\mu} \quad (1)$$

Let  $x = n\lambda$ , the function  $f_2(x)$  can be written as:

$$f_2(n\lambda) = \sqrt{\frac{1 - n\lambda}{1 - \lambda}} + \frac{1 - \lambda}{2(1 - n\lambda)} - 1/2$$

Let  $g(n\lambda) = \lim_{\lambda \rightarrow 0} f_2(n\lambda)$ , we have

$$g(n\lambda) = \sqrt{1 - n\lambda} + \frac{1}{2(1 - n\lambda)} - 1/2$$

Let  $z = \sqrt{1 - n\lambda}$ , we have:

$$g(z) = z + 1/(2z^2) - 1/2$$

The derivative of function  $g(z)$  over  $z$  can be written as:

$$g'(z) = 1 - 1/z^3$$

Given that  $z \in (0, 1)$ , we have  $g'(z) < 0$ . Then  $g(z)$  attains its minimum 1 when  $z$  approaches 1. Equivalently,  $f_2(n\lambda)$  attains its minimum 1 when  $n\lambda$  approaches 0.

Considering Eq.(1), we find that  $n\lambda$  is bounded on  $(0, 1/2)$  if  $\mu < 1/2$ ,

We then calculate the difference between  $f_2(n\lambda)$  and its minimum. It can be written as

$$\begin{aligned}\Delta f_2(n\lambda) &= \lim_{\lambda \rightarrow 0} [f_2(x) - f_2^*] \\ &= g(n\lambda) - 1 \\ &= \sqrt{1 - n\lambda} + \frac{1}{2(1 - n\lambda)} - 3/2\end{aligned}$$

Let  $m = n\lambda$ , the derivative of function  $\Delta f_2(m)$  over  $m$  can be written as:

$$\Delta f_2'(m) = \frac{1 - (1 - m)^{3/2}}{2(1 - m)^2} > 0$$

Then  $\Delta f_2(m)$  is monotonically increasing over  $(0, 1/2)$ . Therefore  $\Delta f_2(m)$  is bounded on  $(0, \sqrt{2}/2 - 1/2)$   $\square$

## 2 Proof of Theorem 2

**Theorem 2.** Suppose we are given two Binomial distributions,  $\mathcal{B}(n, \lambda)$  and  $\mathcal{B}(n, \lambda^0)$  with  $n \rightarrow +\infty$ ,  $\lambda^0 \rightarrow 0$  and  $\lambda \rightarrow 0$ , where  $n$  is increasing while  $\lambda$  and  $\lambda^0$  are decreasing. There exists a real constant  $m$  and another real constant  $m^{(0)}$ , such that if  $m = n\lambda$  and  $m^{(0)} = n\lambda^{(0)}$  and if  $\lambda > \lambda^{(0)}$ , we have:

$$\text{KL}(\mathcal{B}(n, \lambda) || \mathcal{B}(n, \lambda^0)) < m \log \frac{m}{m^{(0)}} + (1 - m) \log \frac{1 - m + m^2/2}{1 - m^{(0)} + m^{(0)2}/2}$$

*Proof.* Let  $m = n\lambda$  and  $m^{(0)} = n\lambda^{(0)}$ , we have

$$\begin{aligned}\text{KL}(\mathcal{B}(n, \lambda) || \mathcal{B}(n, \lambda^{(0)})) &= n\lambda \log \frac{\lambda}{\lambda^{(0)}} + n(1 - \lambda) \log \frac{1 - \lambda}{1 - \lambda^{(0)}} \\ &= n\lambda \log \frac{n\lambda}{n\lambda^{(0)}} + n(1 - \lambda) \log \frac{1 - \lambda}{1 - \lambda^{(0)}} \\ &= m \log \frac{m}{m^{(0)}} + n(1 - \lambda) \log \frac{1 - \lambda}{1 - \lambda^{(0)}}\end{aligned} \quad (2)$$

We then take the right part,

$$g = n(1 - \lambda) \log \frac{1 - \lambda}{1 - \lambda^{(0)}} = (1 - \lambda) \log \frac{(1 - \lambda)^n}{(1 - \lambda^{(0)})^n}$$

By Taylor series' theorem with Lagrange remainder,  $g$  can be written as:

$$g = (1 - \lambda) \log \frac{1 - n\lambda + \frac{n(n-1)}{2}\lambda^2 + R_{j=2}(-\lambda)}{1 - n\lambda^{(0)} + \frac{n(n-1)}{2}\lambda^{(0)2} + R_{j=2}(-\lambda^{(0)})}$$

There exists a  $\theta \in (0, 1)$  such that,

$$R_{j=2}(x) = \frac{x^3(n-2)(n-1)n(1+x\theta)^{n-3}}{6}$$

Given that  $n \rightarrow +\infty$  and  $x \in (-1, 1)$ , we have  $(R)_{j=2}'(x) > 0$ . Therefore,  $R_{j=2}(x)$  is monotonically increasing over  $(-1, 1)$ . Since  $\lambda > \lambda^{(0)}$ , we have

$$R_{j=2}(-\lambda) < R_{j=2}(-\lambda^{(0)}) \quad (3)$$

We then seek to prove:

$$k = \frac{1 - n\lambda + \frac{n(n-1)}{2}\lambda^2}{1 - n\lambda^{(0)} + \frac{n(n-1)}{2}\lambda^{(0)^2}} < 1$$

Let  $f(x) = n(n-1)x^2/2 - nx + 1$ , we have

$$k = \frac{f(\lambda)}{f(\lambda^{(0)})}$$

Here,  $f(x)$  is an U-shaped parabola with axis  $x = 1/(n-1)$ . By theorem 1, we have  $n\lambda < 1/2$ , then we have  $\lambda^0 < \lambda < 1/(n-1)$ , then  $f(x)$  is monotonically increasing over the support of  $\lambda^0$  and  $\lambda$ , namely

$$f(\lambda) < f(\lambda^0) \tag{4}$$

With Eq.(3) and Eq.(4),  $g$  can be written as:

$$\begin{aligned} g &< (1 - \lambda) \log \frac{1 - n\lambda + \frac{n(n-1)}{2}\lambda^2}{1 - n\lambda^{(0)} + \frac{n(n-1)}{2}\lambda^{(0)^2}} \\ &= (1 - \lambda) \log \frac{1 - m + m^2/2 - n\lambda^2/2}{1 - m^{(0)} + m^{(0)^2}/2 - n\lambda^{(0)^2}/2} \end{aligned}$$

Similarly, let  $h(x) = 1 - x + x^2/2$ . It is an U-shaped parabola with axis  $x = 1$  such that

$$\begin{aligned} -n\lambda^2/2 &< -n\lambda^{(0)^2}/2 \\ 1 - m + m^2/2 &< 1 - m^{(0)} + m^{(0)^2}/2 \end{aligned}$$

Then we have

$$\begin{aligned} g &< (1 - \lambda) \log \frac{1 - m + m^2/2}{1 - m^{(0)} + m^{(0)^2}/2} \\ &< (1 - n\lambda) \log \frac{1 - m + m^2/2}{1 - m^{(0)} + m^{(0)^2}/2} \\ &= (1 - m) \log \frac{1 - m + m^2/2}{1 - m^{(0)} + m^{(0)^2}/2} \end{aligned} \tag{5}$$

Combining Eq.(2) and Eq.(5) concludes the proof. □

### 3 Test of Significance

The statistical significance test tool `sc_stats` from National Institute of Standards and Technology (NIST) is used to compare our RTN and the baseline VSRU on CHiME-2 HMM states classification task. The test results find a significant difference in performance between the RTN and the VSRU at the level of  $p < 0.001$ .

### 4 Table of detailed WER (%) on the CHiME-2 test set

We report the detailed WERs as a function of the SNR in CHiME-2 shown in Table 1. For all SNRs, the RTN outperforms other Baseline RNNs including LSTM, SRU by a large margin. It outperforms the state-of-the-art models including VSRU, RRN and RPPU for most SNRs. This suggests that incorporating the relational thinking into speech recognition lends itself to the model's robustness.

Table 1: Detailed WER (%) on the CHiME-2 test set.

Model	-6 dB	-3 dB	0 dB	3 dB	6 dB	9 dB
LSTM [Huang et al., 2019]	42.4	33.5	26.7	21.1	17.3	15.3
SRU [Huang et al., 2019]	42.5	34.0	26.2	22.2	17.4	15.1
RPPU [Huang et al., 2019]	39.9	31.1	24.9	20.3	16.0	<b>13.2</b>
Our SRU [Lei et al., 2017]	42.1	33	26.1	20.7	16.8	15.1
VSRU [Chung et al., 2015]	41.5	32.8	26.2	20.9	16.9	16.1
RRN [Palm et al., 2018]	40.2	32.1	25.9	20.2	16.2	14.0
RTN (Ours)	<b>39.0</b>	<b>30.4</b>	<b>25.4</b>	<b>19.4</b>	<b>15.5</b>	13.8

## 5 More examples of graphs generated by RTN

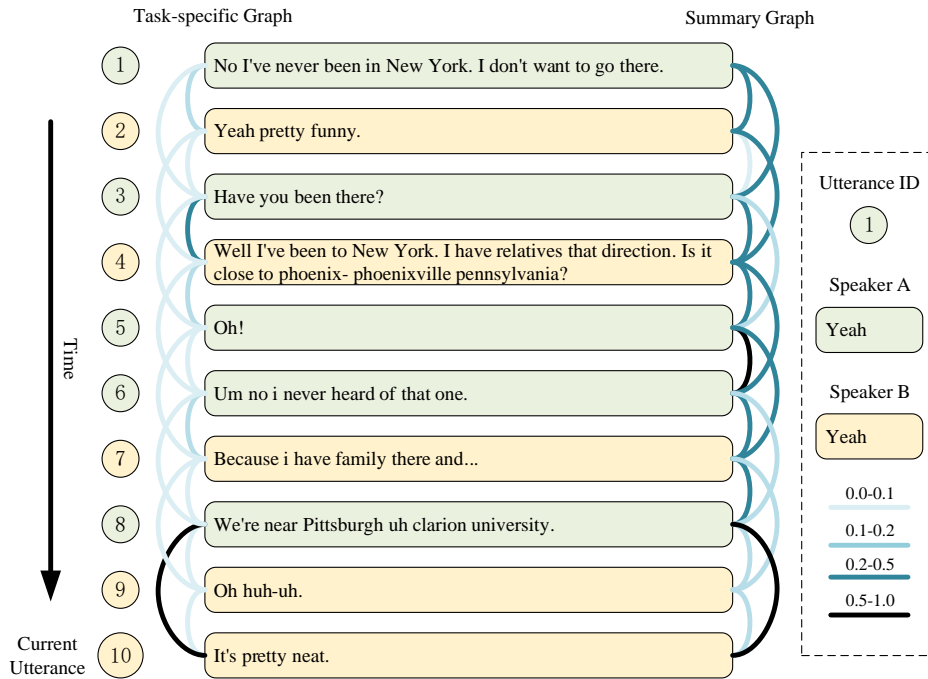


Figure 1: Example of graphs generated by RTN: ten sequential utterances from "sw02262-A\_029098-029769" to "sw02262-B\_031645-031828"

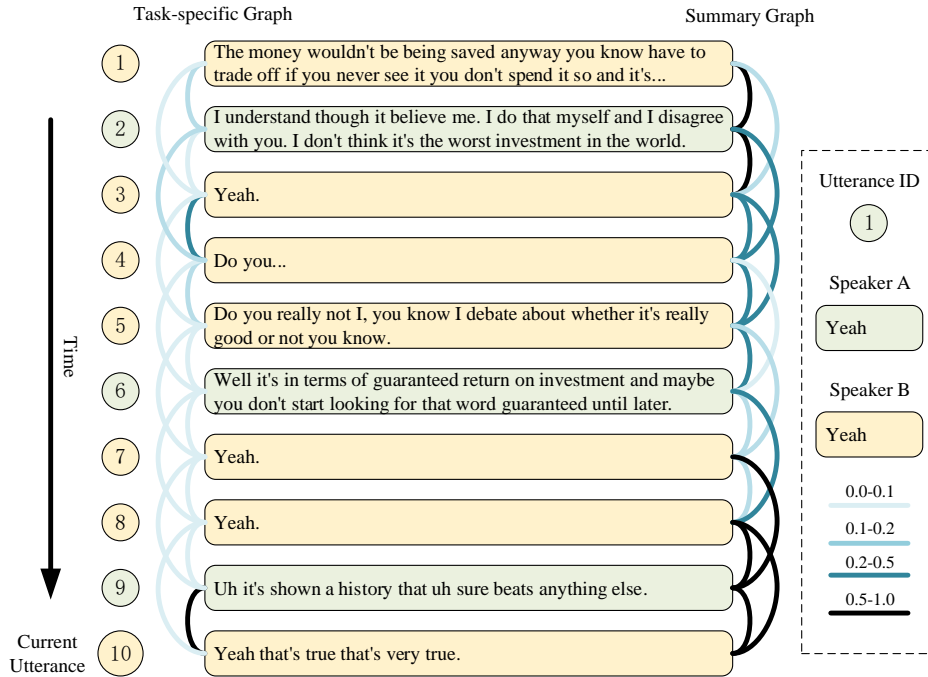


Figure 2: Example of graphs generated by RTN: ten sequential utterances from "sw02062-B\_019277-020062" to "sw02062-B\_022871-023232"

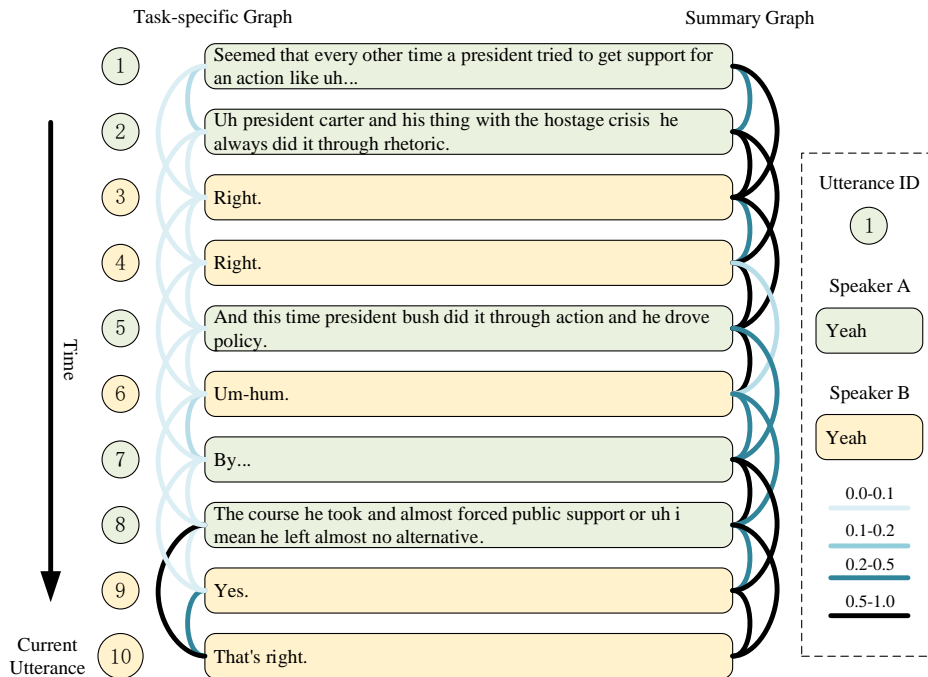


Figure 3: Example of graphs generated by RTN: ten sequential utterances from "sw02130-A\_002749-003357" to "sw02130-B\_005687-005840"

## References

- [Chung et al., 2015] Chung, J., Kastner, K., Dinh, L., Goel, K., Courville, A. C., and Bengio, Y. (2015). A recurrent latent variable model for sequential data. In *Advances in neural information processing systems*, pages 2980–2988.
- [Huang et al., 2019] Huang, H., Wang, H., and Mak, B. (2019). Recurrent poisson process unit for speech recognition. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 6538–6545.
- [Lei et al., 2017] Lei, T., Zhang, Y., Wang, S. I., Dai, H., and Artzi, Y. (2017). Simple recurrent units for highly parallelizable recurrence. *arXiv preprint arXiv:1709.02755*.
- [Palm et al., 2018] Palm, R., Paquet, U., and Winther, O. (2018). Recurrent relational networks. In *Advances in Neural Information Processing Systems*, pages 3368–3378.