WHEN DEEP DENOISING MEETS ITERATIVE PHASE RETRIEVAL

SUPPLEMENTARY MATERIAL

In this supplementary material, we provide proofs on the proximal operators used in our algorithms and show how ADMM [Boyd et al. 2011] with indicator functions coincides with Hybrid-Input-Output (HIO) [Fienup 1982] and Hybrid-Projection-Reflection (HPR) [Bauschke et al. 2003].

1 Proximal operators

We consider two proximal operators for Fourier phase retrieval: the squared error of Fourier amplitudes and regularization by denoising (RED) coupled with additional object-space constraints.

1. \( R(x) = \bar{I}_C(x) + \frac{\lambda}{2} \langle x, x - D(x) \rangle \)

Let \( D \) be the denoiser used in RED and \( C \) be the set of signals satisfying the additional constraints provided, where we assume that the denoiser \( D \) is (locally) homogeneous with symmetric Jacobian [Romano et al. 2017] and \( C \) is a convex set. For any \( \tau > 0 \), if \( v^+ = \text{prox}_{\tau R}(v) \), then the first-order optimality condition gives

\[
\begin{align*}
    v^+ &= \text{argmin}_{x \in \mathbb{R}^n} \tau R(x) + \frac{1}{2} \| v - x \|^2 \\
    \Rightarrow \tau (\partial \bar{I}_C(v^+)) + \lambda (v^+ - D(v^+)) + v^+ - v &= 0 \\
    \Leftrightarrow v^+ &= \left( I + \frac{\tau}{1 + \lambda \tau} \partial \bar{I}_C \right)^{-1} \left( v + \frac{\tau \lambda D(v^+)}{1 + \lambda \tau} \right) \\
    \Leftrightarrow v^+ &= \Pi_C \left( v + \frac{\tau \lambda D(v^+)}{1 + \lambda \tau} \right)
\end{align*}
\]

(S1)

where \( \partial \bar{I}_C \) is the subgradient of the indicator function and the last equality follows by noting that the resolvent of \( \partial \bar{I}_C \) is the projection \( \Pi_C \) onto \( C \) [Ryu & Boyd 2016].

2. \( f(z) = \frac{1}{2} \| y - |Fz| \|^2 \)

Let \( F \) be the (normalized) discrete Fourier transform and \( y \) be the measured Fourier amplitude, which is non-negative. For simplicity, we consider 1D signals only (the conclusion holds for any dimension). Using the overhead symbol \( \hat{\cdot} \) to denote the signal after Fourier transform, Parseval’s theorem gives

\[
\begin{align*}
    x^+ &= \text{prox}_{\tau f}(x) = \text{argmin}_{z} \frac{\tau}{2} \| y - |Fz| \|^2 + \frac{1}{2} \| x - z \|^2 \\
    \Leftrightarrow \hat{x^+} &= \text{argmin}_{\hat{z}} \frac{\tau}{2} \| y - |\hat{z}| \|^2 + \frac{1}{2} \| \hat{x} - \hat{z} \|^2 \\
    &= \text{argmin}_{\hat{z}} \frac{1}{2} \sum_{k} \tau (|\hat{z}[k]| - y[k])^2 + |\hat{x}[k] - \hat{z}[k]|^2
\end{align*}
\]

(S2)

It was noticed in [Wen et al. 2012] that the solution is

\[
\hat{x^+}[k] = \frac{\tau}{\tau + 1} y[k] \frac{\hat{x}[k]}{|\hat{x}[k]|} + \frac{1}{\tau + 1} \hat{z}[k] \quad \forall k
\]

(S3)

which follows from the first-order optimality condition. Here, we provide an alternative proof that this solution is the global minimum.
We start by using the triangle inequality \( |\hat{z}[k] - \hat{x}[k]|^2 \geq (|\hat{z}[k]| - |\hat{x}[k]|)^2 \) to give the lower bound
\[
\min_{\hat{z}} \sum_k \tau(|\hat{z}[k]| - y[k])^2 + |\hat{z}[k] - \hat{x}[k]|^2 \geq \min_{\hat{z}} \sum_k \tau(|\hat{z}[k]| - y[k])^2 + (|\hat{z}[k]| - |\hat{x}[k]|)^2
\]
(S4)
Equality between the right- and left-hand sides is achieved when
\[
\Re(\hat{z}[k]\hat{\bar{z}}[k]) = |\hat{z}[k]| \hat{x}[k] \quad \forall k
\]
(S5)
i.e., when the complex phase \( \angle \hat{z}[k] = \angle \hat{x}[k] \) (\( \angle \hat{z}[k] \) can be arbitrary if \( \hat{x}[k] = 0 \)). As the right-hand side is convex on \( |\hat{z}[k]| \), the minimum is achieved when
\[
|\hat{z}[k]| = \frac{\tau y[k] + |\hat{x}[k]|}{\tau + 1} \quad \forall k
\]
(S6)
as \( y[k], |\hat{x}[k]| \geq 0 \). Therefore, if \( x^+ \) minimizes \( S^2 \), then for all \( k \),
\[
\hat{x}^+[k] = \frac{\tau y[k] + |\hat{x}[k]|}{\tau + 1} \exp(i\angle \hat{x}[k])
\]
(S7)
\[
= \frac{\tau}{\tau + 1} y[k] \exp(i\angle \hat{x}[k]) + \frac{1}{\tau + 1} |\hat{x}[k]| \exp(i\angle \hat{x}[k])
\]
\[
= \frac{\tau}{\tau + 1} y[k] \hat{x}[k] + \frac{1}{\tau + 1} \hat{x}[k]
\]
Performing an inverse Fourier transform gives (26) in the main text:
\[
x^+ = \frac{\tau}{1+\tau} \hat{x} - \frac{1}{\tau + 1} x
\]
(S8)

2 Equivalence between ADMM and HIO/HPR

Let \( x_0 \) be the ground truth and \( S \) and \( \hat{S} \) be the support for \( x_0 \) and the extended support for padded \( \hat{x}_0 = P_{mn}x_0 \), respectively.

If there is additional information about the signal support, e.g. an estimation \( \gamma \) such that \( S \subseteq \gamma \), then the relation \( \hat{S} \subseteq \hat{\gamma} \) holds for the extended support as well. For example, if we use the same vectorization as in the main text, such that
\[
\hat{x} = P_{mn}x = \begin{bmatrix} x \\ 0_{m-n} \end{bmatrix}
\]
(S9)
then we will have \( S = \hat{S} \) and \( \gamma = \hat{\gamma} \). Define subset \( \mathcal{S} \) for the signals satisfying the given support constraint,
\[
\mathcal{S} := \{ x \in \mathbb{C}^n \mid x_i = 0 \ \forall i \notin \gamma \}
\]
(S10)
The projection onto \( \mathcal{S} \) is
\[
\Pi_{\mathcal{S}}(x)_i = \begin{cases} x_i & \text{if } i \in \gamma \\ 0 & \text{otherwise} \end{cases}
\]
(S11)
and similarly for \( \hat{\mathcal{S}} := \{ x \in \mathbb{C}^m \mid x_i = 0 \ \forall i \notin \hat{\gamma} \} \) on the extended support.

According to [Bauschke et al., 2002], HIO with \( \beta = 1 \) can be written as
\[
\hat{x}^{k+1} = \Pi_{\mathcal{S}}(2\Pi_{\mathcal{M}}(\hat{x}^k) - \hat{x}^k) - \Pi_{\mathcal{M}}(\hat{x}^k) + \hat{x}^k
\]
(S12)
We now relate this to the optimization of FPR with the support constraint
\[
\minimize_{z \in \mathbb{C}^n, \hat{x} \in \mathbb{C}^m} I_M(z) + I_{\hat{S}}(x)
\]
(S13)
subject to \( z = O_{mn}x \)

With \( \hat{x} = O_{mn}x \), this can be rewritten as
\[
\minimize_{\hat{x}, z \in \mathbb{C}^m} I_M(z) + I_{\hat{S}}(\hat{x})
\]
(S14)
subject to \( z = \hat{x} \)
for which ADMM gives the update rule as
\[
\tilde{x}^{k+1} = \Pi_S(z^k + u^k)
\]
\[
z^{k+1} = \Pi_M(\tilde{x}^{k+1} - u^k)
\]
\[
u^{k+1} = u^k + z^{k+1} - \tilde{x}^{k+1}
\]
(S15)
As in (Wen et al., 2012), the updates for \(m^{k+1} = \tilde{x}^{k+1} - u^k\) are given by
\[
m^{k+2} = \tilde{x}^{k+2} - u^{k+1}
\]
\[
= \Pi_S(2\Pi_M(m^{k+1}) - m^{k+1}) - \Pi_M(m^{k+1}) + m^{k+1}
\]
(S16)
which coincides with (S12).
Next, we denote \(S_+\) as the set containing signals which not only satisfy the support constraint but also have non-negative elements in the real part:
\[
S_+ := \{ x \in \mathbb{C}^n \mid x_i = 0 \forall i \notin \gamma \text{ and } \Re(x_i) \geq 0 \forall i \}
\]
(S17)
The projection onto \(S_+\) is
\[
\Pi_{S_+}(x) = \Pi_{Re_+}(\Pi_{S}(x))
\]
(S18)
with \(\Pi_{Re_+}\) being the element-wise projection
\[
\Pi_{Re_+}(x)_i = \begin{cases} 
  x_i & \text{if } \Re(x_i) \geq 0 \\
  i \Im(x_i) & \text{otherwise}
\end{cases}
\]
(S19)
Changing \(S\) to \(S_+\) in (S14) and repeating (S15) to (S16) gives the recursion for \(m^{k+1}\) as
\[
m^{k+2} = \Pi_{S_+}(2\Pi_M(m^{k+1}) - m^{k+1}) - \Pi_M(m^{k+1}) + m^{k+1}
\]
(S20)
which coincides with HPR with \(\beta = 1\) (Bauschke et al., 2003).

References